IPA Derivatives for Make-to-Stock Production-Inventory Systems With Backorders Under the (R,r) Policy

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Abstract

This paper addresses *Infinitesimal Perturbation Analysis* (*IPA*) in the class of *Make-to Stock* (*MTS*) production-inventory systems with backorders under the continuous-review (R,r) policy, where R is the stock-up-to level and r is the reorder point. A system from this class is traditionally modeled as a discrete system with discrete demand arrivals at the inventory facility and discrete replenishment orders placed at the production facility. Here, however, we map an underlying discrete MTS system to a *Stochastic Fluid Model* (SFM) counterpart in which stochastic fluid-flow rate processes with piecewise constant sample paths replace the corresponding traditional discrete demand arrival and replenishment stochastic processes, under very mild regularity assumptions. The paper then analyzes the SFM counterpart and derives closed-form IPA derivative formulas of the time-averaged inventory level and time-averaged backorder level with respect to the policy parameters, R and r, and shows them to be unbiased. The obtained formulas are comprehensive in the sense that they are computed for any initial inventory state and any time horizon, and are simple and fast to compute. These properties hold the promise of utilizing IPA derivatives as an ingredient of offline design algorithms and online management and control algorithms of the class of systems under study.

Keywords and Phrases: Infinitesimal Perturbation Analysis (IPA), IPA derivatives, (R,r) policy, Make-to-Stock Production-Inventory System (MTS system), Stochastic Fluid Model (SFM).

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1. Introduction

In this paper, we consider a class of single-stage, single-product *Make-to-Stock* (*MTS*) production-inventory systems with backorders, under the continuous-review (R,r) *policy*, where R is the stock-up-to level and r is the reorder point ($0<r<R$). An MTS system consists of a production facility coupled to an inventory facility: the inventory facility is visited by a stream of demands and the production facility replenishes the inventory facility. The system is driven by random demand and possibly random production processes. We assume that the production facility has an unlimited supply of raw material, so it never starves.

The continuous-review (R,r) policy aims to maintain the *inventory position* (inventory on hand plus replenishments en route minus backorders) at the inventory facility between r and R, but the actual inventory level may initially exceed R or it may drop below r or below zero; we mention that in the fluid-flow model of this paper, the replenishments-en-route component is null. Under the (R,r) policy with backorders, inventory shortfalls are considered *backordered inventory* (essentially, "negative inventory"). This policy utilizes feedback information from the inventory facility to the production facility to modulate replenishment. More specifically, the production facility alternates between two operational states as follows:

- **Busy state:** Replenishment from the production facility is turned on. This state can be in effect initially, or subsequently, after the inventory position drops to or below r, but does not reach or exceed R.
- **Idle state:** Replenishment from the production facility is turned off. This state can be in effect initially, or subsequently, after the inventory position reaches R from below, but does not drop to or below r.

The general description above of an MTS system can be modeled using two related but distinct paradigms: the (traditional) discrete paradigm and the *Stochastic Fluid Model* (*SFM*) paradigm. We mention that MTS systems are usually specified as discrete baseline models, since MTS system descriptions are naturally amenable to such setting. If modeling convenience renders an SFM formulation preferable (as will be discussed later), the discrete baseline model needs to be mapped to an SFM counterpart.

In the discrete MTS model with backorders, demands arrive discretely with some demand size, and the inventory facility attempts to satisfy incoming demands on a first come first serve (FCFS) basis. Demands that cannot be immediately satisfied completely from inventory on hand wait in a FCFS buffer at the inventory facility until sufficient replenishment arrives. Whenever the inventory level drops to or below r (possibly resulting in a shortfall), an order is placed at the production facility so as to raise the inventory position to R. A replenishment corresponding to each order arrives at the inventory facility after a random lead time. Thus, once the inventory position drops to or below R, it is maintained at R thereafter. In contrast, an analytical SFM counterpart has no notion of individual demands, and consequently, no notion of individual lead times and individual replenishments. Rather, an SFM substitutes a stochastic process of *demand arrival rate* for the stochastic process of discrete demand arrivals, and a stochastic process of *inventory replenishment rate* for the stochastic process of replenishment order arrivals. The operational states of the production facility are modulated by inventory-level hitting times: a busy state is inaugurated when the inventory level hits r from above, and an idle state is inaugurated when the inventory level hits R from below. Note that in both a discrete MTS model or its SFM counterpart, the inventory level may exceed R only initially, since once it drops to or below R, it can never exceed R again. A schematic of the SFM is depicted in Figure 1.

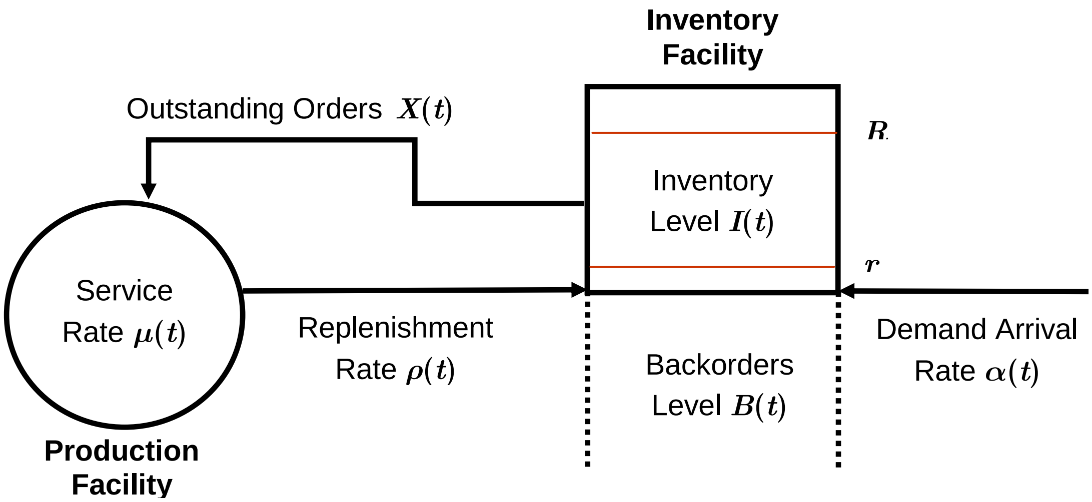

Figure 1. SFM schematic of an MTS production-inventory system with backorders under the (R, r) policy

The subject matter of this paper is *Infinitesimal Perturbation Analysis* (*IPA*) of MTS systems with backorders under the (R, r) policy. IPA is a technique for obtaining sample path derivatives of a random variable $L(\theta)$ with respect to some parameters of interest, θ. For IPA-based applications to be statistically accurate, it is essential that the IPA derivative should be *unbiased* in the sense that the expectation and differentiation operators commute, i.e., $E\left[\frac{d}{d\theta}L(\theta)\right] = \frac{d}{d\theta}E[L(\theta)]$; otherwise, it is said to be *biased*. Sufficient conditions for unbiased IPA derivatives are given in the following result.

Fact 1 (see Rubinstein and Shapiro (1993), Lemma A2, p. 70)
An IPA derivative $\frac{d}{d\theta}L(\theta)$ is unbiased, if

- **(a)** For each θ, the IPA derivatives $\frac{d}{d\theta}L(\theta)$ exist w.p.1 (with probability 1),
- **(b)** W.p.1, $L(\theta)$ is Lipschitz continuous in Θ, and the (random) Lipschitz constants have finite first moments. □

IPA derivatives are nonparametric in the sense that they can be computed from observed data without knowledge of the underlying probability law. Consequently, they can be computed from simulation runs, or in real-life systems deployed in the field, and the values can potentially be used in stochastic optimization. This property holds out the promise of utilizing IPA derivative formulas to provide sensitivity information on system metrics with respect to control parameters of interest, and can serve as the theoretical underpinning for offline design algorithms and online control algorithms. We point out that SFM formulations often enjoy an important advantage over their discrete counterparts: IPA derivatives in SFM setting tend to be unbiased, while their discrete counterparts tend to be biased; two cases in point of bias are Heidelberger et al. (1988) for discrete queueing settings and Fu (1994) for discrete inventory systems under the (s, S)

policy. Comprehensive discussions of IPA derivatives and their applications can be found in Fu and Hu (1997) and Cassandras and Lafortune (1999).

A considerable body of recent work on IPA in queuing context grew out of the observation that IPA derivatives in SFM setting tend to be unbiased. Here, transactions (e.g., packets in a telecommunications transmission system) are modeled as fluid flows, so that random discrete arrivals become random arrival rates and random discrete services become random service rates. Wardi et al. (2002) obtained IPA derivatives in SFM setting for the loss volume and buffer-workload time average with respect to buffer size, a parameter of the arrival rate process and a parameter of the service rate process. Some other representative papers are Liu and Gong (1999), Cassandras et al. (2002), Cassandras et al. (2003), Paschalidis et al. (2004) and Sun et al. (2004). The aforementioned papers obtained IPA derivatives in SFM setting for telecommunication networks, and showed that the IPA derivatives are unbiased and easily computable.

More recently, IPA derivatives were obtained in SFM setting for MTS systems under a continuous-review *base-stock policy* (a special case of the (R,r) policy, where $R=r$). Paschalidis et al. (2004) considered multi-stage MTS systems with backorders in SFM setting, assuming that inventory at each stage is controlled by a continuous-time base-stock policy. The paper computed the right-sided IPA derivatives of the time averaged inventory level and service level with respect to the base-stock level, and then used them to determine the optimal base-stock level at each stage. One limitation of the aforementioned papers (in both queueing and MTS settings) is that they constrain the system to start from a prescribed initial state. In contrast, Zhao and Melamed (2006, 2007) considered any initial inventory state for MTS systems with backorders or lost-sales under the base-stock policy, and derived sided IPA derivative formulas as needed. Specifically, the latter derived IPA derivative formulas for the time averaged inventory level, backorders or lost sales with respect to the base-stock level, as well as a parameter of the production rate process. Finally, these papers also showed that the IPA derivatives are unbiased and easily computable.

In this paper, we consider an SFM formulation of MTS systems with backorders under the continuous-review (R,r) policy. The choice of an SFM for the system is motivated in this case by the fact that the IPA derivatives of the time-averaged inventory level and time-averaged backorder level with respect to the reorder point r are *biased* in the discrete model formulation. More specifically, whenever the inventory position is depleted to the reorder point, a replenishment order is placed; however, if the reorder point is slightly decreased, then no order is placed, thereby engendering a singularity in the IPA derivative with respect to r. Using the SFM formulation, this paper derives the IPA derivative formulas of the time averaged inventory level and backorder level with respect to the stock-up-to level R and the reorder point r, under any initial inventory state. Furthermore the IPA derivatives are shown to be unbiased and easily computable.

Throughout the paper, we use the following notational conventions and terminology. $N(x)$ denotes a neighborhood of x, where x may be vector valued. A function $f(x)$ is said to be *locally differentiable* at x if it is differentiable in a neighborhood of x; it is said to be *locally independent* of x if it is constant in a neighborhood of x. The indicator function of set A is denoted by 1_A, and $x^+ = \max\{x, 0\}$. Finally, all computations of derivatives of integrals make use of the generalized Leibniz integral rule in Lemma A.1 and Corollary A.1 of Appendix A.

The rest of the paper is organized as follows. Section 2 specifies the mapping from discrete MTS models with backorders under the (R,r) policy to SFM counterparts. Section 3 describes the performance metrics and parameters of interest. Section 4 obtains IPA derivative formulas and shows them to be unbiased. Finally, Section 5 concludes the paper, and Appendix A provides some auxiliary material.

2. Mapping a Discrete MTS Model to an SFM Counterpart

We next proceed to map the traditional discrete MTS model with backorders under the (R,r) policy into an SFM counterpart. By convention, all stochastic processes to follow are assumed right-continuous.

- $\{I(t): t \geq 0\}$ is the *inventory level process,* where $I(t)$ is the fluid volume of inventory on hand at time t.

- $\{B(t): t \geq 0\}$ is the *backorder level process,* where $B(t)$ is the fluid volume of all backorders at time t.

- $\{W(t): t \geq 0\}$ is the *extended inventory level process* [see Zhao and Melamed (2006, 2007)], where

$$W(t) = I(t) - B(t) = \begin{cases} I(t), & \text{if } B(t) = 0 \\ -B(t), & \text{if } I(t) = 0 \end{cases} \tag{2.1}$$

  Thus, $W(t)$ determines both $I(t)$ and $B(t)$ (and vice versa).

- $\{X(t): t \geq 0\}$ is the *outstanding orders process,* where $X(t)$ is the fluid volume of outstanding orders at time t.

- $\{\alpha(t): t \geq 0\}$ is the *demand arrival rate process,* where $\alpha(t)$ is the incoming demand arrival rate at time t.

- $\{\mu(t): t \geq 0\}$ is the *production rate process,* where $\mu(t)$ is the production rate (capacity) of the production facility at time t.

- $\{\rho(t): t \geq 0\}$ is the *replenishment rate process,* where $\rho(t)$ is the replenishment rate of product from the production facility to the inventory facility at time t.

We now proceed to exhibit the formal definitions of all fluid-model components of the MTS system under study.

The initial state of the system is given by $(X(0), W(0))$, subject to the constraints $\{W(0) \geq R\} \subset \{X(0) = 0\}$ and $\{W(0) \leq r\} \subset \{X(0) > 0\}$. Furthermore, We shall characterize the production system state at time t by $X(t)$, namely, the state is idle on the event $\{X(t) = 0\}$ and it is busy on $\{X(t) > 0\}$. Given an initial state, $(X(0), W(0))$ and the processes $\{\alpha(t)\}$ and $\{\mu(t)\}$, we define the evolution of the system as follows.

While in an idle state, the inventory level process is governed by the one-sided stochastic differential equation

$$\frac{d}{dt^+} I(t) = \frac{d}{dt^+} W(t) = -\alpha(t), \tag{2.2}$$

and the model satisfies

$$\rho(t) = 0, \tag{2.3}$$

$$B(t) = X(t) = 0. \tag{2.4}$$

While in a busy state, the extended inventory level process is governed by the one-sided stochastic differential equation

$$\frac{d}{dt^+} W(t) = \rho(t) - \alpha(t) = \mu(t) - \alpha(t), \tag{2.5}$$

and the model satisfies the conservation equation

$$X(t) + I(t) - B(t) = R \tag{2.6}$$

and the relationships

$$I(t) = [R - X(t)]^+, \quad B(t) = [X(t) - R]^+. \tag{2.7}$$

Let $[0, T]$ be a finite time interval for some prescribed time horizon, T. Similarly to Wardi et al. (2002), we define two types of sample-path events:

- **Exogenous events.** An exogenous event occurs in sample path ω at time t if t is a jump time, such that $\alpha(\omega, t-) \neq \alpha(\omega, t+)$ or $\mu(\omega, t-) \neq \mu(\omega, t+)$.
- **Endogenous events.** An endogenous event occurs in sample path ω at time t if t is an extended-inventory level hitting time, such that $W(\omega, t) = R$ or $W(\omega, t) = r$ or $W(\omega, t) = 0$.

3. Performance Metrics and Parameters

We shall be interested in the following performance metrics in the mapped SFM, over a finite interval $[0, T]$:

- The time average of fluid volume of inventory on-hand over the interval $[0, T]$, given by

$$M_I(T) = \frac{1}{T} \int_0^T I(t)\, dt. \tag{3.1}$$

- The time average of fluid volume of backorders over the interval $[0, T]$, given by

$$M_B(T) = \frac{1}{T} \int_0^T B(t)\, dt. \tag{3.2}$$

Let $\theta \in \Theta$ denote a generic parameter of interest from a closed and bounded domain Θ. We shall routinely write $R(\theta)$, $r(\theta)$, $M_I(T,\theta)$, $M_B(T,\theta)$ and so on, to explicitly display the dependence of a random variable on its parameters of interest. Thus, our goal is to derive formulas for the IPA derivatives $\frac{\partial}{\partial\theta} M_I(T,\theta)$ and $\frac{\partial}{\partial\theta} M_B(T,\theta)$ in SFM setting, using sample path analysis, and to show them to be unbiased. The two IPA parameters of interest are:

- The stock-up-to level of the inventory facility,

$$R(\theta) = \theta, \quad \theta \in \Theta. \tag{3.3}$$

- The reorder point of the inventory facility,

$$r(\theta) = \theta, \quad \theta \in \Theta. \tag{3.4}$$

4. IPA Derivatives

Define a sequence of intervals $(P_j(\theta), Q_j(\theta))$, $j = 1, ..., J(\theta)$, as the ordered extremal subintervals of $[0,T)$, such that $X(t,\theta) > 0$ for all $t \in (P_j(\theta), Q_j(\theta))$, and $J(\theta)$ is the (random) number of such intervals inaugurated in $[0,T)$; the term extremal means here that the endpoints, $P_j(\theta)$ and $Q_j(\theta)$, are obtained via the inf and sup functions respectively. By convention, if any of these endpoints does not exist, then it is set to ∞. We refer to intervals of the form $[P_j(\theta), Q_j(\theta))$ as *busy intervals*, since the system is in the busy state for their duration. In a similar vein, intervals of the form $[Q_j(\theta), P_{j+1}(\theta))$ will be refereed to as *idle intervals*. Thus, from a certain index and on, $P_j(\theta)$ is an inventory hitting time of r, while $Q_j(\theta)$ is an inventory hitting time of R. We have the following observation for the hitting times $P_j(\theta)$ and $Q_j(\theta)$.

Observation 1
For any finite time interval $[0,T)$, we have w.p.1,

(a) When they exist, $P_1(\theta) < Q_1(\theta) < P_2(\theta) < Q_2(\theta) < ... < P_{J(\theta)}(\theta) < Q_{J(\theta)}(\theta)$.

(b) $P_1(\theta) = 0$ on $\{X(0) > 0\}$ and $P_1(\theta) > 0$ on $\{X(0) = 0\}$.

(c) $\int_0^{P_1(\theta)} \alpha(t)\,dt = W(0) - r$ and $\int_{P_1(\theta)}^{Q_1(\theta)} [\mu(t) - \alpha(t)]\,dt = R - r$ on $\{X(0) = 0\}$.

(d) $\int_0^{Q_1(\theta)} [\mu(t) - \alpha(t)]\,dt = R - W(0)$ on $\{X(0) > 0\}$.

(e) $\int_{P_j(\theta)}^{Q_j(\theta)} [\mu(t) - \alpha(t)]\,dt = R - r$, $j > 1$.

(f) $\int_{Q_j(\theta)}^{P_{j+1}(\theta)} \alpha(t)\,dt = R - r,\ j > 1.$ □

Throughout this chapter, we assume the following regularity conditions.

Assumption 1

(a) W.p.1, the demand arrival rate process, $\{\alpha(t)\}$, and the production rate process, $\{\mu(t)\}$, are piecewise-constant, and have a finite number of discontinuities in any finite time interval, such that the time points at which the discontinuities occur are independent of the parameters of interest.

(b) W.p.1, $\alpha(P_j(\theta)) \ge a^*$, $\mu(P_j(\theta)) - \alpha(P_j(\theta)) \le D^*$, $\mu(Q_j(\theta)) - \alpha(Q_j(\theta)) \ge d^*$ and $\alpha(Q_j(\theta)) \le A^*$, where a^*, D^*, d^*,and A^* are positive deterministic constants independent of $\theta \in \Theta$ and $j \ge 1$.

(c) No multiple sample-path events occur simultaneously w.p.1.

(d) The initial inventory state does not depend on θ, namely, w.p.1, $W(0,\theta) = W(0)$ and $X(0,\theta) = X(0)$, for all $\theta \in \Theta$. □

Assumption 1 implies the following observation.

Observation 2

(a) w.p.1, $J(\theta) \le J^*$, where J^* is a positive deterministic constant, independent of $\theta \in \Theta$.

(b) There is a neighborhood $N(P_j(\theta),\theta)$, such that for all $(t,\theta) \in N(P_j(\theta),\theta)$, $\alpha(t) = \alpha(P_j(\theta)) > 0$ and $\mu(t) - \alpha(t) = \mu(P_j(\theta)) - \alpha(P_j(\theta))$.

(c) There is a neighborhood $N(Q_j(\theta),\theta)$, such that for all $(t,\theta) \in N(Q_j(\theta),\theta)$, $\alpha(t) = \alpha(Q_j(\theta))$ and $\mu(t) - \alpha(t) = \mu(Q_j(\theta)) - \alpha(Q_j(\theta)) > 0$. □

All results in the sequel are to be understood as holding w.p.1.

4.1 IPA Derivatives with Respect to R

To obtain the IPA derivatives of $M_I(T,\theta)$ and $M_B(T,\theta)$ with respect to the stock-up-to level, R, we make the following assumptions.

Assumption 2

(a) $R(\theta) = \theta$, $\theta \in \Theta$.

(b) The processes $\{\alpha(t)\}$ and $\{\mu(t)\}$, and the reorder point r are independent of the parameter $R(\theta) = \theta$. □

Lemma 1

(a) $P_1(\theta)$ is locally independent of θ.

(b) $P_j(\theta)$ and $Q_j(\theta)$, $j \ge 1$, are continuous.

Proof
To prove part (a), we consider two cases.
Case 1: The system starts in busy state, namely, on the event $\{X(0) > 0\}$, whence the result follows from $P_1(\theta) = 0$ by part (b) of Observation 1.
Case 2: The system starts in idle state, namely, on the event $\{X(0) = 0\}$, whence $P_1(\theta) > 0$ by part (b) of Observation 1. The result follows from part (c) of Observation 1, since by part (d) of Assumption 1 and part (b) of Assumption 2, $W(0)$, $\alpha(t)$ and r are all independent of θ.

We prove part (b) by induction, letting $R(\theta) = \theta$ be some given stock-up-to level. Clearly, $P_1(\theta)$ is trivially continuous in θ by part (a), thereby establishing the induction basis for part (b) of this lemma.

To prove the inductive step for $Q_j(\theta)$, assume that $P_j(\theta)$ is continuous by the induction hypothesis. Next subtracting the equation in part (e) of Observation 1 with θ from the same equation with $\theta + \Delta\theta$ yields

$$\begin{aligned}\Delta\theta &= \int_{P_j(\theta+\Delta\theta)}^{Q_j(\theta+\Delta\theta)} [\mu(t) - \alpha(t)]\, dt - \int_{P_j(\theta)}^{Q_j(\theta)} [\mu(t) - \alpha(t)]\, dt \\ &= \int_{Q_j(\theta)}^{Q_j(\theta+\Delta\theta)} [\mu(t) - \alpha(t)]\, dt - \int_{P_j(\theta)}^{P_j(\theta+\Delta\theta)} [\mu(t) - \alpha(t)]\, dt\end{aligned} \tag{4.1}$$

Sending $\Delta\theta \downarrow 0$ in Eq. (4.1) and focusing on the rightmost side above, we note that the second integral tends to zero by the induction hypothesis, and that by part (c) of Observation 2, the integrand is positive and constant in some neighborhoods of $Q_j(\theta)$ and $Q_j(\theta + \Delta\theta)$. Consequently, we conclude that $\lim_{\Delta\theta \downarrow 0} [Q_j(\theta + \Delta\theta) - Q_j(\theta)] = 0$.

Finally, to prove the inductive step for $P_j(\theta)$, assume that $Q_{j-1}(\theta)$ is continuous by the induction hypothesis. Next subtracting the equation in part (f) of Observation 1 with θ from the same equation with $\theta + \Delta\theta$ yields

$$\begin{aligned}\Delta\theta &= \int_{Q_{j-1}(\theta+\Delta\theta)}^{P_j(\theta+\Delta\theta)} \alpha(t)\, dt - \int_{Q_{j-1}(\theta)}^{P_j(\theta)} \alpha(t)\, dt \\ &= \int_{P_j(\theta)}^{P_j(\theta+\Delta\theta)} \alpha(t)\, dt - \int_{Q_{j-1}(\theta)}^{Q_{j-1}(\theta+\Delta\theta)} \alpha(t)\, dt\end{aligned} \tag{4.2}$$

Sending $\Delta\theta \downarrow 0$ in Eq. (4.2) and focusing on the rightmost side above, we note that the second integral tends to zero by the induction hypothesis, and that by part (b) of Observation 2, the integrand is positive and constant in some neighborhoods of $P_j(\theta)$ and $P_j(\theta + \Delta\theta)$. Consequently, we conclude that $\lim_{\Delta\theta \downarrow 0} [P_j(\theta + \Delta\theta) - P_j(\theta)] = 0$. □

Lemma 2
$P_j(\theta)$ and $Q_j(\theta)$, $j \geq 1$, are continuously differentiable.

Proof
We prove the lemma by induction, letting $R(\theta) = \theta$ be some given stock-up-to level. Clearly, $P_1(\theta)$ is continuously differentiable in θ by part (a) of Lemma 1, thereby establishing the induction basis for this lemma.

To prove the inductive step for $Q_j(\theta)$, assume that $P_j(\theta)$ is continuously differentiable by the induction hypothesis. In view of part (a) of Assumption 1 and Lemma 1, Eq. (4.1) can be written w.p.1 after division by a sufficiently small $\Delta\theta$ as

$$\frac{[\mu(Q_j(\theta)) - \alpha(Q_j(\theta))][Q_j(\theta+\Delta\theta) - Q_j(\theta)]}{\Delta\theta} = \frac{[\mu(P_j(\theta)) - \alpha(P_j(\theta))][P_j(\theta+\Delta\theta) - P_j(\theta)] + \Delta\theta}{\Delta\theta} \tag{4.3}$$

Sending $\Delta\theta \downarrow 0$ in Eq. (4.3), we note that by the induction hypothesis, its right-hand side converges and the limit is continuous. Furthermore, since $\mu(Q_j(\theta)) - \alpha(Q_j(\theta)) > 0$ by part (c) of Observation 2, we conclude that $Q_j(\theta)$ is continuously differentiable.

To prove the inductive step for $P_j(\theta)$, assume that $Q_{j-1}(\theta)$ is continuously differentiable by the induction hypothesis. In view of part (a) of Assumption 1 and Lemma 1, Eq. (4.2) can be written w.p.1 after division by a sufficiently small $\Delta\theta$ as

$$\frac{\alpha(P_j(\theta))[P_j(\theta+\Delta\theta) - P_j(\theta)]}{\Delta\theta} = \frac{\alpha(Q_{j-1}(\theta))[Q_{j-1}(\theta+\Delta\theta) - Q_{j-1}(\theta)] + \Delta\theta}{\Delta\theta} \tag{4.4}$$

Sending $\Delta\theta \downarrow 0$ in Eq. (4.4), we note that by the induction hypothesis, its right-hand side converges and the limit is continuous. Furthermore, since $\alpha(P_j(\theta)) > 0$ by part (b) of Observation 2, we conclude that $P_j(\theta)$ is continuously differentiable. □

Lemma 3

$$\frac{d}{d\theta}\int_{P_j(\theta)}^{Q_j(\theta)} [\rho(\tau) - \alpha(\tau)]\, d\tau = \frac{d}{d\theta}\int_{P_j(\theta)}^{Q_j(\theta)} [\mu(\tau) - \alpha(\tau)]\, d\tau = [\mu(Q_j(\theta)) - \alpha(Q_j(\theta))]\frac{d}{d\theta}Q_j(\theta) - [\mu(P_j(\theta)) - \alpha(P_j(\theta))]\frac{d}{d\theta}P_j(\theta)$$

and

$$\frac{d}{d\theta}\int_{Q_j(\theta)}^{P_{j+1}(\theta)}\alpha(\tau)\,d\tau=\alpha(P_{j+1}(\theta))\frac{d}{d\theta}P_{j+1}(\theta)-\alpha(Q_j(\theta))\frac{d}{d\theta}Q_j(\theta)$$

Proof

By Eq. (2.5), parts (a) and (c) of Assumption 1 and part (b) of Assumption 2, we have that $\{\mu(t)\}$ and $\{\alpha(t)\}$ are independent of θ, continuous at the hitting times $P_j(\theta)$ and $Q_j(\theta)$, and have a finite number of discontinuities between successive hitting times. Lemma 3 now follows from Lemma A.1, Corollary A.1, and Lemma 2. □

Lemma 4

Consider a mapped SFM of an MTS system with backorders under the (R,r) policy, subject to Assumption 1 and 2. Then, for any $\theta\in\Theta$, $0<T<\infty$,

$$\frac{d}{d\theta}P_j(\theta)=\begin{cases}0, & j=1\\ \dfrac{1+\alpha(Q_{j-1}(\theta))\frac{d}{d\theta}Q_{j-1}(\theta)}{\alpha(P_j(\theta))}, & j>1\end{cases} \tag{4.5}$$

and

$$\frac{d}{d\theta}Q_j(\theta)=\frac{1+[\mu(P_j(\theta))-\alpha(P_j(\theta))]\,\frac{d}{d\theta}P_j(\theta)}{\mu(Q_j(\theta))-\alpha(Q_j(\theta))},\quad j\geq 1 \tag{4.6}$$

Proof

We first prove the lemma for $j=1$. Eq. (4.5) for $j=1$ follows from part (a) of Lemma 1. To prove Eq. (4.6) for $j=1$, we consider two cases.

Case 1: The system starts in busy state, namely, on the event $\{X(0)>0\}$. It follows that $P_1(\theta)=0$ by part (b) of Observation 1. Next, note that by Eq. (2.5), $\int_0^{Q_1(\theta)}[\mu(\tau)-\alpha(\tau)]\,d\tau=R(\theta)-W(0)$, which on differentiation with respect to θ with the aid of Lemma 3 yields after some manipulation

$$\frac{d}{d\theta}Q_1(\theta)=\frac{1}{\mu(Q_1(\theta))-\alpha(Q_1(\theta))}\quad \text{on } \{X(0)>0\},$$

since $W(0)$, $\mu(t)$ and $\alpha(t)$ are all independent of θ by part (d) of Assumption 1 and part (b) of Assumption 2. Eq. (4.6) for $j=1$ now follows from the equation above, since $\mu(Q_1(\theta))>\alpha(Q_1(\theta))\geq 0$ by part (c) of Observation 2.

Case 2: The system starts in idle state, namely, on the event $\{X(0)=0\}$, whence $P_1(\theta)>0$. Note that by part (e) of Observation 1, $\int_{P_1(\theta)}^{Q_1(\theta)}[\mu(\tau)-\alpha(\tau)]\,d\tau=R(\theta)-r$, which on differentiation with respect to θ with the aid of Lemma 3 and Eq. (4.5), yields after some manipulation

$$\frac{d}{d\theta}Q_1(\theta)=\frac{1}{\mu(Q_1(\theta))-\alpha(Q_1(\theta))} \quad \text{on } \{X(0)=0\},$$

since $W(0)$, $\{\alpha(t)\}$, $\{\mu(t)\}$ and r are all independent of θ by part (d) of Assumption 1 and part (b) of Assumption 2. Eq. (4.6) now follows for $j=1$ from the equation above, since $\frac{d}{d\theta}P_1(\theta)=0$ and $\mu(Q_1(\theta))>\alpha(Q_1(\theta))\geq 0$ by part (a) of Lemma 1 and part (c) of Observation 2, respectively.

Next, we prove Eq. (4.5) for $j>1$. By part (f) of Observation 1 we have $\int_{Q_{j-1}(\theta)}^{P_j(\theta)}\alpha(\tau)\,d\tau=R(\theta)-r$, which on differentiation with respect to θ with the aid of Lemma 3 yields

$$\alpha(P_j(\theta))\frac{d}{d\theta}P_j(\theta)-\alpha(Q_{j-1}(\theta))\frac{d}{d\theta}Q_{j-1}(\theta)=1,$$

and after some manipulation the above becomes

$$\frac{d}{d\theta}P_j(\theta)=\frac{1+\alpha(Q_{j-1}(\theta))\frac{d}{d\theta}Q_{j-1}(\theta)}{\alpha(P_j(\theta))}, \quad j>1.$$

Eq. (4.5) for $j>1$ now follows from the equation above, since $\alpha(P_j(\theta))>0$ by part (b) of Observation 2.

Finally, we prove Eq. (4.6) for $j>1$. By part (e) of Observation 1 we have $\int_{P_j(\theta)}^{Q_j(\theta)}[\mu(\tau)-\alpha(\tau)]\,d\tau=R(\theta)-r$, $j>1$, which on differentiation with respect to θ with the aid of Lemma 3 yields

$$\left[\mu(Q_j(\theta))-\alpha(Q_j(\theta))\right]\frac{d}{d\theta}Q_j(\theta)-\left[\mu(P_j(\theta))-\alpha(P_j(\theta))\right]\frac{d}{d\theta}P_j(\theta)=1, \quad j>1,$$

and after some manipulation the above becomes

$$\frac{d}{d\theta}Q_j(\theta)=\frac{1+\left[\mu(P_j(\theta))-\alpha(P_j(\theta))\right]\frac{d}{d\theta}P_j(\theta)}{\mu(Q_j(\theta))-\alpha(Q_j(\theta))}, \quad j>1.$$

Eq. (4.6) for $j>1$ now follows from the equation above, since $\mu(Q_j(\theta))>\alpha(Q_j(\theta))\geq 0$ by part (c) of Observation 2. □

Lemma 5

Consider a mapped SFM of an MTS system with backorders under the (R,r) policy, subject to Assumption 1 and 2. Then for any $\theta\in\Theta$, $0<T<\infty$, and $t\in[0,T]$,

(a) On the event $A(\theta)=\{0<t<Q_1(\theta)\}$,

$$\frac{\partial}{\partial\theta}I(t,\theta)=0, \tag{4.7}$$

$$\frac{\partial}{\partial\theta} B(t,\theta) = 0. \tag{4.8}$$

(b) On the events $B_j(\theta) = \{Q_j(\theta) < t < P_{j+1}(\theta)\},\ j \ge 1$,

$$\frac{\partial}{\partial\theta} I(t,\theta) = 1 + \alpha(Q_j(\theta))\ \frac{d}{d\theta} Q_j(\theta), \tag{4.9}$$

$$\frac{\partial}{\partial\theta} B(t,\theta) = 0. \tag{4.10}$$

(c) On the events $C_j(\theta) = \{P_j(\theta) < t < Q_j(\theta)\} \cap \{I(t,\theta) > 0\},\ j \ge 2$,

$$\frac{\partial}{\partial\theta} I(t,\theta) = -[\mu(P_j(\theta)) - \alpha(P_j(\theta))]\ \frac{d}{d\theta} P_j(\theta), \tag{4.11}$$

$$\frac{\partial}{\partial\theta} B(t,\theta) = 0. \tag{4.12}$$

(d) On the events $D_j(\theta) = \{P_j(\theta) < t < Q_j(\theta)\} \cap \{B(t,\theta) > 0\},\ j \ge 2$,

$$\frac{\partial}{\partial\theta} I(t,\theta) = 0, \tag{4.13}$$

$$\frac{\partial}{\partial\theta} B(t,\theta) = [\mu(P_j(\theta)) - \alpha(P_j(\theta))]\ \frac{d}{d\theta} P_j(\theta). \tag{4.14}$$

Proof
To prove part (a), we consider two cases.

Case 1: The system starts in busy state, namely, on the event $\{X(0) > 0\}$, so $P_1(\theta) = 0$ by part (b) of Observation 1. Next, note that by Eq. (2.5), $W(t,\theta) = W(0) + \int_0^t [\mu(\tau) - \alpha(\tau)]\, d\tau$ on the event $A(\theta) \cap \{X(0) > 0\}$. By part (d) of Assumption 1 and part (b) of Assumption 2, we have $W(0)$, $\{\alpha(t)\}$ and $\{\mu(t)\}$ are all independent of θ, implying that $W(t,\theta)$ is independent of θ on $A(\theta) \cap \{X(0) > 0\}$. It follows from Eq. (2.1) that both $I(t,\theta)$ and $B(t,\theta)$ are also independent of θ on $A(\theta) \cap \{X(0) > 0\}$, from which Eqs. (4.7) and (4.8) follow by Eq. (2.1).

Case 2: The system starts in idle state, namely, on the event $\{X(0) = 0\}$, so $P_1(\theta) > 0$ by part (b) of Observation 1. Since $A(\theta) = \{0 < t < P_1(\theta)\} \cup \{P_1(\theta) \le t < Q_1(\theta)\}$, we shall treat each of the constituent events separately.

On the event $\{0 < t < P_1(\theta)\} \cap \{X(0) = 0\}$, $W(t,\theta) = W(0) - \int_0^t \alpha(\tau)\, d\tau$ by Eq. (2.2). Since $W(0)$ and $\{\alpha(t)\}$ are independent of θ, it follows that $W(t,\theta)$ is independent of θ on $\{0 < t < P_1(\theta)\} \cap \{X(0) = 0\}$. Eqs. (4.7) and (4.8) now follow from Eq. (2.1).

On the event $\{P_1(\theta) \le t < Q_1(\theta)\} \cap \{X(0) = 0\}$, the system is in the busy state, so $W(t,\theta) = r + \int_{P_1(\theta)}^{t} [\mu(\tau) - \alpha(\tau)]\, d\tau$ by Eq. (2.5). Since $P_1(\theta)$, r, $\{\alpha(t)\}$ and $\{\mu(t)\}$ are all independent of θ by part (a) of Lemma 1 and part (b) of Assumption 2, it follows that $W(t,\theta)$ is independent of θ on $\{P_1(\theta) \le t < Q_1(\theta)\} \cap \{X(0) = 0\}$. Eqs. (4.7) and (4.8) now follow from Eq. (2.1).

To prove part (b), note that by Eq. (2.2) we have on each event $B_j(\theta)$, $I(t,\theta) = R(\theta) - \int_{Q_j(\theta)}^{t} \alpha(\tau)\, d\tau$ and $B(t,\theta) = 0$, which on differentiation with respect to θ yield Eqs. (4.9) and (4.10), respectively.

To prove parts (c) and (d), note that by Eq. (2.5) we have on each event $C_j(\theta) \cup D_j(\theta)$, $W(t,\theta) = r + \int_{P_j(\theta)}^{t} [\mu(\tau) - \alpha(\tau)]\, d\tau$, which on differentiation with respect to θ yields on $C_j(\theta) \cup D_j(\theta)$

$$\frac{d}{d\theta} W(t,\theta) = -[\mu(P_j(\theta)) - \alpha(P_j(\theta))] \frac{d}{d\theta} P_j(\theta).$$

On $C_j(\theta)$ one has $I(t,\theta) = W(t,\theta)$ and $B(t,\theta) = 0$, while on $D_j(\theta)$, one has $I(t,\theta) = 0$ and $B(t,\theta) = -W(t,\theta)$, both by Eq. (2.1), from which parts (c) and (d) follow. Note that the events $\bigcup_{j=2}^{J(\theta)} \{P_j(\theta) < t < Q_j(\theta)\} \cap \{I(t,\theta) = B(t,\theta) = 0\}$ need not be considered, because they have probability zero by part (c) of Assumption 1. □

Theorem 1
Consider a mapped SFM of an MTS system with backorders under the (R, r) policy, subject to Assumptions 1 and 2. Then for any $\theta \in \Theta$, $0 < T < \infty$, and $t \in [0, T]$, the IPA derivatives of the inventory time average and the backorder time average with respect to the stock-up-to level parameter, R, are given by

$$\frac{\partial}{\partial\theta} M_I(T,\theta) = \frac{1}{T} \sum_{j=1}^{J(\theta)} \left[1 + \alpha(Q_j(\theta)) \frac{d}{d\theta} Q_j(\theta)\right] \left[\min\{P_{j+1}(\theta), T\} - Q_j(\theta)\right] - \frac{1}{T} \sum_{j=2}^{J(\theta)} \left[\mu(P_j(\theta)) - \alpha(P_j(\theta))\right] \frac{d}{d\theta} P_j(\theta) \int_{P_j(\theta)}^{\min\{Q_j(\theta), T\}} 1_{\{I(t,\theta) > 0\}} dt \quad (4.15)$$

and

$$\frac{\partial}{\partial\theta} M_B(T,\theta) = \frac{1}{T} \sum_{j=2}^{J(\theta)} \left[\mu(P_j(\theta)) - \alpha(P_j(\theta))\right] \frac{d}{d\theta} P_j(\theta) \int_{P_j(\theta)}^{\min\{Q_j(\theta), T\}} 1_{\{B(t,\theta) > 0\}} dt, \quad (4.16)$$

where $\frac{d}{d\theta}P_j(\theta)$ and $\frac{d}{d\theta}Q_j(\theta)$ are given by Eqs. (4.5) and (4.6), respectively.
Furthermore, the IPA derivatives in Eqs. (4.15) and (4.16) are unbiased for each finite $T > 0$ and every $\theta \in \Theta$.

Proof
Applying the generalized Leibniz integral rule to Eqs. (3.1) and (3.2) yields

$$\frac{\partial}{\partial\theta}M_I(T,\theta) = \frac{1}{T}\frac{d}{d\theta}\int_0^T I(t,\theta)\,dt = \frac{1}{T}\int_0^T \frac{\partial}{\partial\theta}I(t,\theta)\,dt. \tag{4.17}$$

$$\frac{\partial}{\partial\theta}M_B(T,\theta) = \frac{1}{T}\frac{d}{d\theta}\int_0^T B(t,\theta)\,dt = \frac{1}{T}\int_0^T \frac{\partial}{\partial\theta}B(t,\theta)\,dt. \tag{4.18}$$

To see that, note that the end-points of the interval $[0,T]$ do not depend on θ, while the hitting times in the interior of $[0,T]$ satisfy by the continuity of $\{I(t,\theta)\}$ and $\{B(t,\theta)\}$,

$$\begin{aligned}
I(P_j(\theta)-,\theta)\frac{d}{d\theta}P_j(\theta) &= I(P_j(\theta)+,\theta)\frac{d}{d\theta}P_j(\theta)\\
B(P_j(\theta)-,\theta)\frac{d}{d\theta}P_j(\theta) &= B(P_j(\theta)+,\theta)\frac{d}{d\theta}P_j(\theta)\\
I(Q_j(\theta)-,\theta)\frac{d}{d\theta}Q_j(\theta) &= I(Q_j(\theta)+,\theta)\frac{d}{d\theta}Q_j(\theta)\\
B(Q_j(\theta)-,\theta)\frac{d}{d\theta}Q_j(\theta) &= B(Q_j(\theta)+,\theta)\frac{d}{d\theta}Q_j(\theta)
\end{aligned}$$

Consequently, Eqs. (4.17) and (4.18) follow by rearranging the terms on the right-hand sum in Corollary A.1. Eqs. (4.15) and (4.16) now follow by substituting the values of the derivatives computed in Lemma 5 into Eqs. (4.17) and (4.18).

We next prove that the IPA derivative formulas are unbiased using Fact 1. The proofs of Eqs. (4.17) and (4.18) establish that Condition (a) of Fact 1 is satisfied for both $M_I(T,\theta)$ and $M_B(T,\theta)$ for each finite $T > 0$. By parts (a) and (b) of Assumption 1 and part (a) of Observation 2, Eqs. (4.15) and (4.16) are bounded in θ w.p.1. Since any differentiable function with such a bounded derivative is Lipschitz continuous and its Lipschitz constant trivially has a finite first moment, it follows that Condition (b) of Fact 1 holds for both $M_I(T,\theta)$ and $M_B(T,\theta)$ for each finite $T > 0$, thereby completing the proof. □

4.2 IPA Derivatives with Respect to r

To obtain the IPA derivatives of $M_I(T,\theta)$ and $M_B(T,\theta)$ with respect to the reorder point, r, we make the following assumptions.

Assumption 3

(a) $r(\theta) = \theta$, where $\theta \in \Theta$.

(b) The processes $\{\alpha(t)\}$ and $\{\mu(t)\}$, and the stock-up-to level, R, are independent of the parameter θ. □

Lemma 6
The hitting times $P_j(\theta)$ and $Q_j(\theta)$, $j \geq 1$, are continuous.

Proof
Similar to that of Lemma 1. □

Lemma 7
The hitting times $P_j(\theta)$ and $Q_j(\theta)$, $j \geq 1$, are continuously differentiable.

Proof
Similar to that of Lemma 2. □

Lemma 8

$$\frac{d}{d\theta}\int_{P_j(\theta)}^{Q_j(\theta)}[\rho(\tau)-\alpha(\tau)]\,d\tau = \frac{d}{d\theta}\int_{P_j(\theta)}^{Q_j(\theta)}[\mu(\tau)-\alpha(\tau)]\,d\tau =$$
$$[\mu(Q_j(\theta))-\alpha(Q_j(\theta))]\frac{d}{d\theta}Q_j(\theta)-[\mu(P_j(\theta))-\alpha(P_j(\theta))]\frac{d}{d\theta}P_j(\theta)$$

and

$$\frac{d}{d\theta}\int_{Q_j(\theta)}^{P_{j+1}(\theta)}\alpha(\tau)\,d\tau = \alpha(P_{j+1}(\theta))\frac{d}{d\theta}P_{j+1}(\theta)-\alpha(Q_j(\theta))\frac{d}{d\theta}Q_j(\theta)$$

Proof
Similar to that of Lemma 3. □

Lemma 9
Consider a mapped SFM of an MTS system with backorders under the (R, r) policy, subject to Assumption 1 and 3. Then, for any $\theta \in \Theta$, $0 < T < \infty$,

$$\frac{d}{d\theta}P_j(\theta) = \begin{cases} 0, & j=1 \text{ and } X(0)>0 \\ \dfrac{-1}{\alpha(P_1(\theta))}, & j=1 \text{ and } X(0)=0 \\ \dfrac{\alpha(Q_{j-1}(\theta))\frac{d}{d\theta}Q_{j-1}(\theta)-1}{\alpha(P_j(\theta))}, & j>1 \end{cases} \tag{4.19}$$

and

$$\frac{d}{d\theta}Q_j(\theta) = \begin{cases} 0, & j=1 \text{ and } X(0)>0 \\ \dfrac{[\mu(P_j(\theta))-\alpha(P_j(\theta))]\frac{d}{d\theta}P_j(\theta)-1}{\mu(Q_j(\theta))-\alpha(Q_j(\theta))}, & \text{otherwise} \end{cases} \tag{4.20}$$

Proof
We first prove Eq. (4.19) for $j=1$. We consider two cases.

Case 1: The system starts in busy state, namely, on the event $\{X(0)>0\}$. It follows that $P_1(\theta)=0$, which implies the first line of Eq. (4.19).

Case 2: The system starts in idle state, namely, on the event $\{X(0)=0\}$, whence $P_1(\theta)>0$. Note that by part (c) of Observation 1 one has $\int_0^{P_1(\theta)} \alpha(t)\,dt = W(0)-r(\theta)$, which on differentiation with respect to θ yields after some manipulation

$$\frac{d}{d\theta}P_1(\theta)=\frac{-1}{\alpha(P_1(\theta))} \quad \text{on } \{X(0)=0\},$$

since $W(0)$ and $\{\alpha(t)\}$ are all independent of θ by part (d) of Assumption 1 and part (b) of Assumption 3, respectively. The second line of Eq. (4.19) now follows from the equation above, since $\alpha(P_1(\theta))>0$ by part (b) of Assumption 1.

Next, we prove Eq. (4.19) for $j>1$. By part (f) of Observation 1, one has $\int_{Q_{j-1}(\theta)}^{P_j(\theta)} \alpha(\tau)\,d\tau = R-r(\theta)$, which on differentiation with respect to θ yields

$$\alpha(P_j(\theta))\frac{d}{d\theta}P_j(\theta)-\alpha(Q_{j-1}(\theta))\frac{d}{d\theta}Q_{j-1}(\theta)=-1,$$

and after some manipulation the above becomes

$$\frac{d}{d\theta}P_j(\theta)=\frac{\alpha(Q_{j-1}(\theta))\frac{d}{d\theta}Q_{j-1}(\theta)-1}{\alpha(P_j(\theta))}, \quad j>1.$$

Eq. (4.19) for $j>1$ now follows from the equation above, since $\alpha(P_j(\theta))>0$ by part (b) of Assumption 1.

Finally, we prove Eq. (4.20). We first consider $j=1$ on the event $\{X(0)>0\}$, and note that by part (d) of Observation 1, $\int_0^{Q_1(\theta)} [\mu(\tau)-\alpha(\tau)]\,d\tau = R-W(0)$. Since $W(0)$, $\{\alpha(t)\}$, $\{\mu(t)\}$ and R are all independent of θ by part (d) of Assumption 1 and part (b) of Assumption 3, the first line of Eq. (4.20) now follows. Otherwise, for $j=1$ on the event $\{X(0)=0\}$ or for $j>1$, one has by part (c) and (e) of Observation 1, $\int_{P_j(\theta)}^{Q_j(\theta)} [\mu(\tau)-\alpha(\tau)]\,d\tau = R-r(\theta)$, which on differentiation with respect to θ yields

$$\left[\mu(Q_j(\theta))-\alpha(Q_j(\theta))\right]\frac{d}{d\theta}Q_j(\theta)-\left[\mu(P_j(\theta))-\alpha(P_j(\theta))\right]\frac{d}{d\theta}P_j(\theta)=-1,$$

and after some manipulation the above becomes

$$\frac{d}{d\theta}Q_j(\theta)=\frac{\left[\mu(P_j(\theta))-\alpha(P_j(\theta))\right]\frac{d}{d\theta}P_j(\theta)-1}{\mu(Q_j(\theta))-\alpha(Q_j(\theta))}.$$

The second line of Eq. (4.20) now follows from the equation above, since $\mu(Q_j(\theta)) > \alpha(Q_j(\theta)) \geq 0$ by part (b) of Assumption 1. □

Lemma 10

Consider a mapped SFM of an MTS system with backorders under the (R, r) policy, subject to Assumption 1 and 3. Then for any $\theta \in \Theta$, $0 < T < \infty$, and $t \in [0, T]$,

(a) On the events

$A_1(\theta) = \{0 < t < P_1(\theta)\} \cap \{X(0) = 0\}$ and $A_2(\theta) = \{0 < t < P_2(\theta)\} \cap \{X(0) > 0\}$

$$\frac{\partial}{\partial\theta} I(t, \theta) = 0, \tag{4.21}$$

$$\frac{\partial}{\partial\theta} B(t, \theta) = 0. \tag{4.22}$$

(b) On the events $B_1(\theta) = \{P_1(\theta) < t < Q_1(\theta)\} \cap \{X(0) = 0\} \cap \{I(t, \theta) > 0\}$ and $B_2(\theta) = \left[\bigcup_{j=2}^{J(\theta)} \{P_j(\theta) < t < Q_j(\theta)\}\right] \cap \{I(t, \theta) > 0\}$,

$$\frac{\partial}{\partial\theta} I(t, \theta) = 1 - [\mu(P_j(\theta)) - \alpha(P_j(\theta))] \frac{d}{d\theta} P_j(\theta), \tag{4.23}$$

$$\frac{\partial}{\partial\theta} B(t, \theta) = 0. \tag{4.24}$$

(c) On the events $C_1(\theta) = \{P_1(\theta) < t < Q_1(\theta)\} \cap \{X(0) = 0\} \cap \{B(t, \theta) > 0\}$ and $C_2(\theta) = \left[\bigcup_{j=2}^{J(\theta)} \{P_j(\theta) < t < Q_j(\theta)\}\right] \cap \{B(t, \theta) > 0\}$,

$$\frac{\partial}{\partial\theta} I(t, \theta) = 0, \tag{4.25}$$

$$\frac{\partial}{\partial\theta} B(t, \theta) = [\mu(P_j(\theta)) - \alpha(P_j(\theta))] \frac{d}{d\theta} P_j(\theta) - 1. \tag{4.26}$$

(d) On the events

$D_1(\theta) = \{Q_1(\theta) < t < P_2(\theta)\} \cap \{X(0) = 0\}$ and $D_2(\theta) = \bigcup_{j=2}^{J(\theta)} \{Q_j(\theta) < t < P_{j+1}(\theta)\}$,

$$\frac{\partial}{\partial\theta} I(t, \theta) = \alpha(Q_j(\theta)) \frac{d}{d\theta} Q_j(\theta), \tag{4.27}$$

$$\frac{\partial}{\partial\theta} B(t, \theta) = 0. \tag{4.28}$$

Proof

To prove part (a) on $A_1(\theta)$, note that the system starts in the idle state, implying $P_1(\theta) > 0$. Next, note that by Eq. (2.2), $W(t, \theta) = W(0) - \int_0^t \alpha(\tau)\, d\tau$ on $A_1(\theta)$. By part (d) of

Assumption 1 and part (b) of Assumption 3, we have that $W(0)$ and $\{\alpha(t)\}$ are independent of θ, respectively, implying that $W(t,\theta)$ is independent of θ on $A_1(\theta)$. It follows from Eq. (2.1) that both $I(t,\theta)$ and $B(t,\theta)$ are also independent of θ on $A_1(\theta)$, from which Eqs. (4.21) and (4.22) follow on $A_1(\theta)$.

To prove part (a) on $A_2(\theta)$, note that the system starts in busy state, implying $P_1(\theta)=0$. Next, decompose $A_2(\theta)=A_{2,1}(\theta)\cup A_{2,2}(\theta)$, where $A_{2,1}(\theta)=\{0<t<Q_1(\theta)\}\cap\{X(0)>0\}$ and $A_{2,2}(\theta)=\{Q_1(\theta)\le t<P_2(\theta)\}\cap\{X(0)>0\}$, and treat each of the constituent events separately.

On $A_{2,1}(\theta)$ one has $W(t,\theta)=W(0)+\int_0^t[\mu(\tau)-\alpha(\tau)]\,d\tau$ by Eq. (2.5). Since $W(0)$, $\{\alpha(t)\}$ and $\{\mu(t)\}$ are all independent of θ by part (d) of Assumption 1 and part (b) of Assumption 3, it follows that $W(t,\theta)$ is independent of θ on $A_{2,1}(\theta)$. It further follows from Eq. (2.1) that both $I(t,\theta)$ and $B(t,\theta)$ are also independent of θ on $A_{2,1}(\theta)$, from which Eqs. (4.21) and (4.22) follow on $A_{2,1}(\theta)$.

On $A_{2,2}(\theta)$ one has $W(t,\theta)=R-\int_{Q_1(\theta)}^t\alpha(\tau)\,d\tau$ by Eq. (2.2). Since $Q_1(\theta)$, R and $\{\alpha(t)\}$ are all independent of θ by Eq. (4.20) and part (b) of Assumption 3, it follows that $W(t,\theta)$ is independent of θ on $A_{2,2}(\theta)$. It further follows from Eq. (2.1) that both $I(t,\theta)$ and $B(t,\theta)$ are also independent of θ on $A_{2,2}(\theta)$, from which Eqs. (4.21) and (4.22) follow on $A_{2,2}(\theta)$.

To prove parts (b) and (c), note that on the event $B_1(\theta)\cup B_2(\theta)\cup C_1(\theta)\cup C_2(\theta)$ one has by Eq. (2.5), $W(t,\theta)=r(\theta)+\int_{P_j(\theta)}^t[\mu(\tau)-\alpha(\tau)]\,d\tau$, which on differentiation with respect to θ yields on $B_1(\theta)\cup B_2(\theta)\cup C_1(\theta)\cup C_2(\theta)$

$$\frac{\partial}{\partial\theta}W(t,\theta)=1-[\mu(P_j(\theta))-\alpha(P_j(\theta))]\frac{d}{d\theta}P_j(\theta).$$

On $B_1(\theta)\cup B_2(\theta)$ one has $I(t,\theta)=W(t,\theta)$ and $B(t,\theta)=0$, while on $C_1(\theta)\cup C_2(\theta)$, one has $I(t,\theta)=0$ and $B(t,\theta)=-W(t,\theta)$, both by Eq. (2.1), from which parts (b) and (c) follow. Note that the events $\left[\{P_1(\theta)<t<Q_1(\theta)\}\cap\{X(0)=0\}\right]\cap\{I(t,\theta)=B(t,\theta)=0\}$ and $\bigcup_{j=2}^{J(\theta)}\{P_j(\theta)<t<Q_j(\theta)\}\cap\{I(t,\theta)=B(t,\theta)=0\}$ need not be considered, because they have probability zero by part (c) of Assumption 1.

Finally, to prove part (d), note that on the event $D_1(\theta)\cup D_2(\theta)$ one has by Eq. (2.2) $I(t,\theta)=R-\int_{Q_j(\theta)}^{t}\alpha(\tau)\,d\tau$ and $B(t,\theta)=0$, which on differentiation with respect to θ yield Eqs. (4.27) and (4.28), respectively. □

Theorem 2

Consider a mapped SFM of an MTS system with backorders under the (R,r) policy, subject to Assumptions 1 and 3. Then for any $\theta\in\Theta$, $0<T<\infty$, and $t\in[0,T]$, the IPA derivatives of the inventory time average and the backorder time average with respect to the reorder point parameter, r, are given by

$$\begin{aligned}\frac{\partial}{\partial\theta}M_I(T,\theta)=&\\ &1_{\{X(0)=0\}}\left(\frac{1}{T}\left[1-[\mu(P_1(\theta))-\alpha(P_1(\theta))]\frac{d}{d\theta}P_1(\theta)\right]\int_{\min\{P_1(\theta),T\}}^{\min\{Q_1(\theta),T\}}1_{\{I(t,\theta)>0\}}dt+\right.\\ &\left.\frac{1}{T}\alpha(Q_1(\theta))\frac{d}{d\theta}Q_1(\theta)\left[\min\{P_2(\theta),T\}-\min\{Q_1(\theta),T\}\right]\right)+\\ &\frac{1}{T}\sum_{j=2}^{J(\theta)}\alpha(Q_j(\theta))\frac{d}{d\theta}Q_j(\theta)\left[\min\{P_{j+1}(\theta),T\}-\min\{Q_j(\theta),T\}\right]+\\ &\frac{1}{T}\sum_{j=2}^{J(\theta)}\left[1-[\mu(P_j(\theta))-\alpha(P_j(\theta))]\frac{d}{d\theta}P_j(\theta)\right]\int_{P_j(\theta)}^{\min\{Q_j(\theta),T\}}1_{\{I(t,\theta)>0\}}dt\end{aligned}\tag{4.29}$$

and

$$\begin{aligned}\frac{\partial}{\partial\theta}M_B(T,\theta)=&\\ &1_{\{X(0)=0\}}\frac{1}{T}\left[[\mu(P_1(\theta))-\alpha(P_1(\theta))]\frac{d}{d\theta}P_1(\theta)-1\right]\int_{\min\{P_1(\theta),T\}}^{\min\{Q_1(\theta),T\}}1_{\{B(t,\theta)>0\}}dt+\\ &\frac{1}{T}\sum_{j=2}^{J(\theta)}\left[[\mu(P_j(\theta))-\alpha(P_j(\theta))]\frac{d}{d\theta}P_j(\theta)-1\right]\int_{P_j(\theta)}^{\min\{Q_j(\theta),T\}}1_{\{B(t,\theta)>0\}}dt\end{aligned}\tag{4.30}$$

where $\frac{d}{d\theta}P_j(\theta)$ and $\frac{d}{d\theta}Q_j(\theta)$ are given by Eqs. (4.19) and (4.20), respectively.

Furthermore, the IPA derivatives in Eqs. (4.29) and (4.30) are unbiased for each finite $T>0$ and every $\theta\in\Theta$.

Proof.

Applying the generalized Leibniz integral rule to Eqs. (3.1) and (3.2) and using an argument similar to that in the proof of Theorem 1 yields

$$\frac{\partial}{\partial\theta}M_I(T,\theta)=\frac{1}{T}\frac{d}{d\theta}\int_0^T I(t,\theta)\,dt=\frac{1}{T}\int_0^T\frac{\partial}{\partial\theta}I(t,\theta)\,dt.\tag{4.31}$$

$$\frac{\partial}{\partial\theta}M_B(T,\theta)=\frac{1}{T}\frac{d}{d\theta}\int_0^T B(t,\theta)\,dt=\frac{1}{T}\int_0^T\frac{\partial}{\partial\theta}B(t,\theta)\,dt.\tag{4.32}$$

Eqs. (4.29) and (4.30) now follow by substituting the values of the derivatives computed in Lemma 10 into Eqs. (4.31) and (4.32).

We next prove that the IPA derivative formulas are unbiased using Fact 1. The proofs of Eqs. (4.29) and (4.30) establish that Condition (a) of Fact 1 is satisfied for both $M_I(T,\theta)$ and $M_B(T,\theta)$ for each finite $T > 0$. By parts (a) and (b) of Assumption 1 and part (a) of Observation 2, Eqs. (4.29) and (4.30) are bounded in θ w.p.1. Since any differentiable function with such a bounded derivative is Lipschitz continuous and its Lipschitz constant trivially has a finite first moment, it follows that Condition (b) of Fact 1 holds for both $M_I(T,\theta)$ and $M_B(T,\theta)$ for each finite $T > 0$, thereby completing the proof. □

5. Conclusion

This paper treated an SFM model of an MTS system with backorders under the continues-review (R,r) policy. In addition to deriving the IPA derivative formulas of the time averaged inventory level and backorder level with respect to the stock-up-to level, R, and the reorder point, r, under any initial inventory state, the IPA derivatives were shown to be unbiased and easily computable.

Future research directions include both theoretical and practical topics. On the theoretical side, the SFM can be used to obtain IPA derivatives in more general supply chains with different policies, or with multiple stages or multiple products. On the practical side, it would be of interest to investigate the efficacy of SFM IPA derivatives as an ingredient in offline design algorithms and online control algorithms of MTS systems and their extensions.

Appendix A

This Appendix provides a generalized Leibniz integral rule, which extends the classical Leibniz Integral Rule below.

Fact 2 (see Fikhtengolts (1965),Volume II, pp. 145-146)
Let $f(t,\theta)$ be defined on the rectangle $[A,B]\times[\varphi,\psi]$, and let $a(\theta)$ and $b(\theta)$ be two differentiable functions over $[\varphi,\psi]$, satisfying $A \le a(\theta) \le b(\theta) \le B$ for each $\theta \in [\varphi,\psi]$. Let $D = \{(t,\theta) : \theta \in [\varphi,\psi], t \in [a(\theta), b(\theta)]\}$. Suppose that $f(t,\theta)$ is continuous on D with a continuously differentiable partial derivative $\frac{\partial}{\partial\theta} f(t,\theta)$ on D. Then the function

$$F(\theta) = \int_{a(\theta)}^{b(\theta)} f(t,\theta)\,dt$$

is differentiable on $[\varphi,\psi]$, and its derivative is given by

$$\frac{d}{d\theta}F(\theta) = \frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(t,\theta)\,dt =$$
$$\int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(t,\theta)\,dt + f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta)$$

□

We need a slight generalization of the Leibnitz Integral rule for the case, where $f(t,\theta)$ is allowed to be discontinuous in t at a finite number of points of each interval $[a(\theta),b(\theta)]$, including its end points.

Lemma A.1 **(**Generalized Leibnitz Integral Rule**)**
Let $f(t,\theta)$ be defined on the rectangle $[A,B]\times[\varphi,\psi]$ and let $a(\theta)$ and $b(\theta)$ be two differentiable functions over $[\varphi,\psi]$, satisfying $A\le a(\theta)\le b(\theta)\le B$ for each $\theta\in[\varphi,\psi]$. Let $D_1=\{(t,\theta):\theta\in[\varphi,\psi],t\in(a(\theta),b(\theta))\}$ and $D_2=\{(t,\theta):\theta\in[\varphi,\psi],t\in[a(\theta),b(\theta)]\}$. Suppose that $f(t,\theta)$ is continuous on D_1 with a continuously differentiable partial derivative $\frac{\partial}{\partial\theta}f(t,\theta)$ on D_1. Suppose further that there exists a function $g(t,\theta)$, such that

1. $g(t,\theta)$ is continuous on D_2 with a continuously differentiable partial derivative $\frac{\partial}{\partial\theta}g(t,\theta)$ on D_2.
2. For every $\theta\in[\varphi,\psi]$, $f(t,\theta)=g(t,\theta)$ on $(a(\theta),b(\theta))$.

Then the function

$$F(\theta)=\int_{a(\theta)}^{b(\theta)}f(t,\theta)\,dt$$

is differentiable on $[\varphi,\psi]$, and its derivative is given by

$$\begin{aligned}\frac{d}{d\theta}F(\theta)&=\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)}f(t,\theta)\,dt=\\&\int_{a(\theta)}^{b(\theta)}\frac{\partial}{\partial\theta}f(t,\theta)\,dt+f(b(\theta)-,\theta)\frac{d}{d\theta}b(\theta)-f(a(\theta)+,\theta)\frac{d}{d\theta}a(\theta)=\\&\int_{a(\theta)}^{b(\theta)}\frac{\partial}{\partial\theta}g(t,\theta)\,dt+g(b(\theta),\theta)\frac{d}{d\theta}b(\theta)-g(a(\theta),\theta)\frac{d}{d\theta}a(\theta)\end{aligned}$$

Proof

If the continuity of $f(t,\theta)$ and $\frac{\partial}{\partial\theta}f(t,\theta)$ holds at the end points $a(\theta)$ and $b(\theta)$, then the Lemma holds trivially by applying the classical Leibnitz Integral Rule to $f(t,\theta)$. Otherwise, let $G(\theta)=\int_{a(\theta)}^{b(\theta)}g(t,\theta)\,dt$, and note that $g(t,\theta)$ satisfies the classical Leibniz Integral Rule. Clearly, $F(\theta)=G(\theta)$ for every $\theta\in[\varphi,\psi]$, since the integrands differ in value only in a finite number of points. Furthermore, the assumptions on $f(t,\theta)$ and $g(t,\theta)$ imply that $g(a(\theta),\theta)=f(a(\theta)+,\theta)$ and $g(b(\theta),\theta)=f(b(\theta)-,\theta)$. The Lemma now follows by applying the classical Leibniz Integral Rule to $g(t,\theta)$. □

Corollary A.1

Let the function $f(t,\theta)$ be defined in the rectangle $[A,B]\times[\varphi,\psi]$, and let $a_i(\theta)$, $1\le i\le n$, be differentiable functions on $[\varphi,\psi]$, satisfying $A\le a_1(\theta)\le a_2(\theta)\le\cdots\le a_n(\theta)\le B$ for every $\theta\in[\varphi,\psi]$. For $1\le i\le n-1$, let $D_{1,i}=\{(t,\theta):\theta\in[\varphi,\psi],t\in(a_i(\theta),a_{i+1}(\theta))\}$ and $D_{2,i}=\{(t,\theta):\theta\in[\varphi,\psi],t\in[a_i(\theta),a_{i+1}(\theta)]\}$. Suppose that for each $1\le i\le n-1$, $f(t,\theta)$ is continuous on $D_{1,i}$ with a continuously differentiable partial derivative $\frac{\partial}{\partial\theta}f(t,\theta)$ on $D_{1,i}$. Suppose further that there exist functions $g_i(t,\theta)$, $1\le i\le n-1$, such that

1. For each $1\le i\le n-1$, $g_i(t,\theta)$ is continuous on $D_{2,i}$ with a continuously differentiable partial derivative $\frac{\partial}{\partial\theta}g_i(t,\theta)$ on $D_{2,i}$.
2. For every $\theta\in[\varphi,\psi]$, $f(t,\theta)=g_i(t,\theta)$ on each $(a_i(\theta),a_{i+1}(\theta))$, $1\le i\le n-1$.

Then the function

$$F(\theta)=\int_{a_1(\theta)}^{a_n(\theta)}f(t,\theta)\,dt$$

is differentiable on $[\varphi,\psi]$, and its derivative is given by

$$\frac{d}{d\theta}F(\theta)=\frac{d}{d\theta}\int_{a_1(\theta)}^{a_n(\theta)}f(t,\theta)\,dt=$$

$$\int_{a_1(\theta)}^{a_n(\theta)}\frac{\partial}{\partial\theta}f(t,\theta)\,dt+\sum_{i=1}^{n-1}\left[f(a_{i+1}(\theta)-,\theta)\frac{d}{d\theta}a_{i+1}(\theta)-f(a_i(\theta)+,\theta)\frac{d}{d\theta}a_i(\theta)\right]$$

Proof

Just decompose the integral into

$$F(\theta)=\int_{a_1(\theta)}^{a_n(\theta)}f(t,\theta)\,dt=\sum_{i=1}^{n-1}\int_{a_i(\theta)}^{a_{i+1}(\theta)}f(t,\theta)\,dt,$$

and apply Lemma A.1 to each term on the right. □

References

[1]. Cassandras, C.G. and S. Lafortune (1999) *Introduction to Discrete Event Systems*, Kluwer Academic Publishers, Boston, MA.

[2]. Cassandras, C.G., Y. Wardi, B. Melamed, G. Sun and C. Panayiotou (2002) "Perturbation Analysis for On-Line Control and Optimization of Stochastic Fluid Models", *IEEE Trans. On Automatic Control*, 47, 1234–1248.

[3]. Cassandras, C.G., G. Sun, C. Panayiotou and Y. Wardi (2003) "Perturbation analysis and control of two-class stochastic fluid models for communication networks," *IEEE Transactions on Automatic Control,* Vol. 48, No. 5, 770–782.

[4]. Fikhtengolts, G.M. (1965) *The Fundamentals of Mathematical Analysis*, Pergamon Press, London.

[5]. Fu, M.C. (1994). "Sample path derivatives for (s,S) inventory systems", *Operations Research*, Vol. 42, 351–364.

[6]. Fu, M.C. and J.Q. Hu (1997) *Conditional Monte Carlo: Gradient Estimation and Optimization Applications,* Kluwer Academic Publishers, Boston, MA.

[7]. Heidelberger, H., X.R. Cao, M. Zazanis and R. Suri (1988) "Convergence properties of infinitesimal analysis estimates", *Management Science,* Vol. 34, 1281–1302.

[8]. Liu, Y. and W.B. Gong (1999) "Perturbation Analysis for Stochastic Fluid Queueing System", *Proceedings of 38th IEEE Conference on Decision Control (CDC),* 4440–4445.

[9]. Paschalidis, I., Y. Liu, C.G. Cassandras and C. Panayiotou (2004) "Inventory Control for Supply Chains with Service Level Constraints: A Synergy between Large Deviations and Perturbation Analysis", *Annals of Operations Research*, Vol. 126, 231–258.

[10]. Rubinstein, R.Y. and A. Shapiro (1993) *Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method*. John Wiley and Sons, New York, NY.

[11]. Sun, G., C. Cassandras and C. Panayiotou, (2004) "Perturbation analysis of multiclass stochastic fluid models," *Discrete Event Dynamic Systems*, Vol. 14, No. 3, 267–307.

[12]. Wardi, Y., B. Melamed, C. G. Cassandras and C. Panayiotou (2002) "On-line IPA gradient estimators in stochastic continuous fluid models", *J. of Optimization Theory and Applications,* Vol. 115, No. 2, 369–405.

[13]. Zhao, Y. and B. Melamed (2006) "IPA Gradients for Make-to-Stock Production-Inventory Systems with Backorders", *Methodology and Computing in Applied Probability* **8**: 191–222.

[14]. Zhao, Y. and B. Melamed (2007) "IPA Gradients for Make-to-Stock Production-Inventory Systems with Lost Sales", *IEEE Transactions on Automatic Control*, Vol. 52, No. 8, 1491–1495.